

## Pigeonhole Principle

### Concepts

1. Pigeonhole Principle gives us a guarantee on what can happen in the worst case scenario. The generalized principle says if  $N$  objects are placed into  $k$  boxes, then at least one box contains at least  $\lceil N/k \rceil$  objects.

### Examples

2. I have 7 pairs of socks in my drawer, one of each color of the rainbow. How many socks do I have to draw out in order to guarantee that I have grabbed at least one pair? What if there are likewise colored pairs of gloves in there and I cannot tell the difference between gloves and socks and I want a matching set?

**Solution:** After grabbing 7 socks, worst case scenario, I have grabbed a sock of each color. Thus, after grabbing one more sock, it has to match up with one of the previous socks so after grabbing 8 socks I am guaranteed to have a pair.

For the second part, after grabbing 21 objects, it is possible that I have grabbed 3 items for each color and hence have gotten no sets yet. But the 22nd thing I grab must complete one of these 7 sets so after 22 items, I am guaranteed to have a matching set.

3. Show that in a class of 30 students in 10B (consisting of freshmen, sophomores, juniors, and seniors), there exists at least 10 freshmen, 8 sophomores, 8 juniors, or 7 seniors.

**Solution:** Since  $30 > (10 - 1) + (8 - 1) + (8 - 1) + (7 - 1) = 29$ , by the generalized pigeonhole principle, one of these things must be true.

### Problems

4. True **FALSE** The Pigeonhole Principle tells us that if we have  $n + 1$  pigeons and  $n$  holes, since  $n + 1 > n$ , each box will have at least one pigeon.

**Solution:** One hole could have all  $n + 1$  pigeons.

5. True **FALSE** The Pigeonhole Principle tells us that with  $n$  pigeons and  $k$  holes each hole can have at most  $\lceil n/k \rceil$  pigeons.

**Solution:** There exists one box with at least that many, but it could contain more.

6. Show that in a  $8 \times 8$  grid, it is impossible to place 9 rooks so that they all don't threaten each other.

**Solution:** By Pigeonhole, there exists one row with at least two rooks, so they must threaten each other.

7. The population of the US is 300 million. Every person has written somewhere between 0 and 10 million lines of code. What's the maximum number of people that we can say must have written the same number of lines of code?

**Solution:** There are  $10 \cdot 10^6 + 1$  different number of lines of code you can write. So, there exists a number of line of codes with at least  $\lceil 300 \cdot 10^6 / (10^6 + 1) \rceil = 30$  people.

8. Three people are running for student government. There are 202 people who vote. What is the minimum number of votes needed for someone to win the election?

**Solution:** By pigeonhole, there exists a person who has gotten at least  $\lceil 202/3 \rceil = 68$  votes. So, someone could win with a  $67 - 67 - 68$  split.

9. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, what is the minimum number of different rooms that will be needed?

**Solution:** There exists a time period will have at least  $\lceil 677/38 \rceil = 18$  classes during it. So 18 different rooms will be needed.

10. Assuming that everything in the US (300 million people) identifies with male or female and has less than 10 children, show that there exist at least 3 people that have the same gender, number of children, three letter initials, and birthday.

**Solution:** We take

$$\left\lceil \frac{300 \cdot 10^6}{2 \cdot 10 \cdot 26^3 \cdot 366} \right\rceil = 3$$

and apply pigeonhole.

11. (Challenge) Show that in a group of 20 people and friendship is mutual, show that there exist two people who have the same number of friends?

**Solution:** Each person can have 0 to 19 friends. But if someone has 0 friends, then no one can have 19 friends and similarly you cannot have 19 friends and no friends. So, there are only 19 options for the number of friends and 20 people, so we can use pigeonhole.

## Permutations and Combinations

### Concepts

12. Permutations are when the order in which we choose matters (e.g. we line people up). The formula for choosing  $k$  things out of a total of  $n$  is  $P(n, k) = n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}$ .

Combinations are when the order does not matter (e.g. choose people for a team). The formula for choosing  $k$  things out of a total of  $n$  is  $C(n, k) = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$ .

### Examples

13. How many ways are there for 8 men and 5 women to stand in a line so that no two women stand next to each other?

**Solution:** Once you place the men, there are 9 spots for the women. We can choose one for each woman to stand in and since the order matters, the final number is  $P(8, 8) \cdot P(9, 5)$ .

14. How many anagrams of MISSISSIPPI exist?

**Solution:** There are a total of 11 letters with 1 M, 2 P, 4 S, 4 I. The final answer is

$$\frac{11!}{2!4!4!}$$

15. How many anagrams of BEAD exist so that the vowels appear all next to each other?

**Solution:** We can group them together at first to get  $(4 - 1)! = 6$  different ways to anagram. But then inside the block of vowels, we can arrange two ways so  $2 \cdot 6 = 12$ .

## Problems

16. **TRUE** False  $P(n, k) = C(n, k) \cdot k!$

**Solution:** You can think of this as choosing  $k$  people in order is the same as first choosing who these  $k$  people are and then arranging them in  $k!$  ways.

17. True **FALSE**  $P(n, k) = P(n, n - k)$ .

**Solution:** This is false but  $C(n, k) = C(n, n - k)$ .

18. How many anagrams of ROYZHAO exist so that the consonants appear next to each other (Y is a vowel)?

**Solution:** 4 vowels and 3 consonants. Block the consonants together to get a total of 5 blocks with two repeating (2 Os) so  $5!/2!$  different ways to arrange. Within the block, there are  $3!$  ways to arrange giving a total of

$$5!/2! \cdot 3! = 360.$$

19. How many ways are there to choose a delegation out of 10 males and 10 females if the delegation is made up of 2 males and 3 females?

**Solution:**  $\binom{10}{2}$  ways to choose the males and  $\binom{10}{3}$  ways to choose the females giving a total of  $\binom{10}{2}\binom{10}{3}$  total ways.

20. At a consultant mixer with 42 people, everyone shakes everyone else's hand exactly once. How many handshakes occur?

**Solution:** There exists one handshake between any two people, so one for each pair. There are  $C(42, 2)$  different ways to choose pairs.

21. 3 different friends are splitting 9 different donuts amongst themselves equally so each person gets 3. How many ways are there to do this?

**Solution:** There are  $C(9, 3)$  different ways to choose donuts for the first person, then  $C(6, 3)$  for the next, and  $C(3, 3)$  for the last. So a total of  $C(9, 3) \cdot C(6, 3) \cdot C(3, 3)$  different ways. Another way to get this is to note that this is the same number of anagrams of AAABBBCCC where the for example in the anagram ABCABCABC, donut numbers 1,4, 7 are given to A. There are a total of  $9!/(3!)^3$  different ways to do this, which is the same answer.

22. How many four digit numbers exist such that their digits are in strictly increasing order?

**Solution:** With any selection of 4 digits from 1 through 9, we can make such a four digit number and every four digit number is made that way. So there are a total of  $C(9, 4)$  different ways.

23. How many rectangle sub-boards with at least two rows and columns exist on a  $8 \times 8$  chessboard?

**Solution:** In order to draw a rectangle, we need to specify the left and right columns, as well as the top and bottom rows. So, we just need to choose two different rows which can happen  $\binom{8}{2}^2$  different ways.

24. (Challenge) There are 9 points on a circle and lines connect all pairs of points. At how many places inside the circle do these lines intersect?

**Solution:** Each intersection inside is determined by four points on the outside. Each selection of 4 points gives a unique intersection point. Thus, there are a total of  $C(9, 4)$  different intersection points.